Distributed Coordination of Multi-Agent Systems With Quantized-Observer Based Encoding-Decoding

Tao Li, *Member, IEEE*, and Lihua Xie, *Fellow, IEEE*

***Abstract*—Integrative design of communication mechanism and coordinated control law is an interesting and important problem for multi-agent networks. In this paper, we consider distributed coordination of discrete-time second-order multi-agent systems with partially measurable state and a limited communication data rate. A quantized-observer based encoding-decoding scheme is designed, which integrates the state observation with encoding/decoding. A distributed coordinated control law is proposed for each agent which is given in terms of the states of its encoder and decoders. It is shown that for a connected network, 2-bit quantizers suffice for the exponential asymptotic synchronization of the states of the agents. The selection of controller parameters and the performance limit are discussed. It is shown that the algebraic connectivity and the spectral radius of the Laplacian matrix of the communication graph play key roles in the closed-loop performance. The spectral radius of the Laplacian matrix is related to the selection of control gains, while the algebraic connectivity is related to the spectral radius of the closed-loop state matrix. Furthermore, it is shown that as the number of agents increases, the asymptotic convergence rate can be approximated as a function of the number of agents, the number of quantization levels (communication data rate) and the ratio of the algebraic connectivity to the spectral radius of the Laplacian matrix of the communication graph.**

***Index Terms*—Data rate, digital communication, distributed coordination, encoding and decoding, multi-agent systems, quantized observer.**

I. Introduction

In recent years, distributed cooperative control of multi-agent systems has attracted unprecedented attention of the control community ([1]–[14]) in view of its wide applications in many emerging fields such as smart grids, intelligent transportation, formation flight, etc. In particular, the problem of multi-agent consensus has been the focus of many researches; see, e.g., [5] and the reference therein.

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T. Li is with the Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China (e-mail: litao@amss.ac.cn).

L. Xie is with EXQUISITUS, Centre for E-City, School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798 (e-mail: elhxie@ntu.edu.sg).

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Quantized consensus is an important problem due to that digital communications are widely adopted and has attracted recurring interest ([15]–[24]). Kashyap *et al.* ([15]) developed an average-consensus algorithm with integer-valued states, which can ensure the asymptotic convergence of agents' states to an integer approximation of the average of the initial states. They gave an upper bound for the expected convergence time for fully connected networks and linear networks. Frasca *et al.* ([19]), Carli *et al.* ([20]), and Li *et al.* ([24]) considered the average-consensus problem with real-valued states and quantized communications. In [19] and [20], static uniform quantizers and dynamic logarithmic quantizers with an infinite number of quantization levels were considered, respectively. In [20] and [24], average-consensus algorithms with dynamic finite-level uniform quantizers were proposed. Especially, in [24], it is shown that if the network is connected, then the control parameters can be properly chosen such that the average-consensus can be achieved with an exponential convergence rate by using a single-bit quantizer. The work of [24] was extended to the cases with link failures in [25] and time-delay in [26], respectively.

The aforementioned works are concerned with the first-order integrator systems with measurable states. In many applications, however, we encounter higher order systems with partially measurable states. Dynamic output feedback control of multi-agent systems of general higher order dynamics was first studied by Fax and Murray ([3]). Tuna proposed a controller design algorithm for synchronization of discrete-time linear systems based on static relative output feedback ([27]). Qu *et al.* ([28]) dealt with static output feedback of multi-agent systems via feedback linearization, where the control input of an agent is given in terms of its own output and the relative output errors with respect to its neighbors. Li *et al.* ([29]) and You and Xie ([30]) considered distributed coordination based on dynamic relative output feedback. Hong *et al.* ([31]) developed a distributed observer for leader-following systems where the leader and the followers are described by second-order integrators and each follower constructs a state observer based on the leader's position, neighbors' positions and leader's control input to estimate the leader's velocity. More literature on distributed observers can be found in [32] and [33].

In this paper, we consider distributed coordination of multi-agent networks based on digital communications. The communications among agents are described by an undirected graph. Each agent is described by a discrete-time second-order integrator, with measurable position but unmeasurable velocity, unlike [20] and [24]. Since the states of the agents are only partially measurable, the encoding-decoding scheme in [24] can not be easily extended to this case. Further, unlike [20] where infinite-level logarithmic quantizers are considered, we aim to design an efficient encoding-decoding scheme under a limited data rate for information exchange between agents. Our first challenge is to jointly design state-observation and encoding-decoding for communication and computation efficiency while achieving consensus. Note that one natural idea is to design a state-observer for each agent and then encode and transmit the state-estimate to neighbors, which, however, requires a distributed control with complex encoding-decoding scheme in order to eliminate the effect of quantization and estimation errors on the final closed-loop system. Further, even such a control scheme can be developed to guarantee convergence, the computation and communication loads are generally higher and the performance (i.e., the convergence rate under the same bit rate) is not definitely better.

From the perspective of minimizing communication bit rate and reducing computation load, we propose an integrative approach for observer and encoder-decoder design in this paper. At each time instant, the quantized innovation of each agent's position is sent to its neighbors, while, at each receiver, an observer-based decoder is activated to obtain an estimate of the sender's position and velocity. Our design can result in a much lower communication requirement due to: 1) the encoder inputs, i.e., agents' positions, contains less variables than the full states; 2) the encoder outputs are in fact a kind of quantized innovations of agents' positions and it is known that innovations generally can be quantized with much lower numbers of bits than the positions themselves. It is worth pointing out that even if the quantization is ignored, our encoders and decoders are different from the dynamic feedback control law in [3]. Here, we do not design a state observer for each agent separately, but send the quantized innovation of each agent's output directly and integrate the state observation and communication process together. Our observer-based encoding-decoding scheme is also different from the distributed observer given in [31], especially, we do not require the knowledge of the other agents' control inputs.

We develop a distributed coordinated control law by using the states of the decoders and encoders, provide sufficient conditions on the control gains and network topology for the existence of finite-level quantizers to ensure the closed-loop convergence, and show that these conditions are also necessary in some sense. We prove that, by selecting the number of quantization levels (data rate) properly, the asymptotic synchronization of the positions and velocities can be achieved. Furthermore, for a connected network, we can always select the control gains, such that 2-bit quantizers can guarantee the exponential convergence of the closed-loop system and the convergence rate can be predesigned.

It should be noted that compared with classical non-quantized and centralized state observers, due to the nonlinearity of the quantization and the coupling of all agents' states, the convergence of a given observer-based encoding-decoding scheme depends on the control inputs of all agents and the closed-loop dynamics of the whole network. Different from [24], the relationship between the estimation error and the quantization error does not have a simple form if observer type is not properly selected, and it is very difficult to get an explicit expression for the relationship between the spectral radius of the closed-loop state matrix and the eigenvalues of the graph Laplacian. All these significantly complicate the closed-loop analysis and the control parameter selection. Also, different from [24], there is no explicit relationship between the stability margin and the control gain, which makes the performance limit analysis difficult. By using differential calculus and limit analysis, we give a linear approximation of the spectral radius of the closed-loop state matrix with respect to the control gain ratio and algebraic connectivity of the communication graph, based on which, a relationship between the performance limit and the parameters of the network and system is revealed. We show that as the number of agents increases to infinity, the asymptotic highest convergence rate is $O(\exp\{-MQ_N^2 t/\sqrt{N}\})$ when using a $(2M+1)$-level quantizer, where Q_N is the ratio of the algebraic connectivity to the spectral radius of the Laplacian matrix of the communication graph.

The remainder of this paper is organized as follows. In Section II, we present the model of the network and agents, give the structures of observer-based encoders, observer-based decoders and distributed coordinated control laws. In Section III, we analyze the closed-loop system and give conditions on the network topology, the control gains and the number of quantization levels to ensure convergence. In Section IV, we discuss the selection of the control gain ratio and show that 2-bit quantizers can guarantee the convergence of the closed-loop system by selecting the control gains properly. We also give an explicit form of the asymptotic convergence rate. In Section V, we draw some concluding remarks and propose future research topics.

The following notation will be used throughout this paper: $\mathbf{1}$ denotes a column vector with all ones. I denotes the identity matrix with an appropriate size. For a given set $\mathcal{S}$, the number of its elements is denoted by $|\mathcal{S}|$. For a given vector or matrix A, we denote its transpose by A^T, its ∞-norm by $\|A\|_\infty$, its Euclidean norm by $\|A\|$, its spectral radius by $\rho(A)$, and its trace by $\mathbf{tr}(A)$. For a given positive number x, the natural logarithm, the logarithm of x with base 2, the maximum integer less than or equal to x, and the minimum integer greater than or equal to x are respectively denoted by $\ln(x)$, $\log_2(x)$, $\lfloor x \rfloor$ and $\lceil x \rceil$.

II. Problem Formulation

A. Agent and Network Models

We consider distributed coordination of a network of agents with the second-order dynamics:

$$\begin{cases} p_i(t+1) = p_i(t) + v_i(t) \\ v_i(t+1) = v_i(t) + u_i(t) \\ z_i(t) = p_i(t),\ t = 0, 1, \ldots \end{cases} \tag{1}$$

where $p_i(t) \in \mathbb{R}$, $v_i(t) \in \mathbb{R}$, and $u_i(t) \in \mathbb{R}$ are the position, velocity and control input of the ith agent, respectively. Here, $z_i(t)$ is the output of agent i, that is, for agent i, only its position is measurable. The agents communicate with each other through a network whose topology is modeled as an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, where the agents and the communication channels between agents are represented by the node set $\mathcal{V}$ and the edge set $\mathcal{E}$, respectively. The weighted adjacency matrix of

$\mathcal{G}$ is denoted by $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$. Note that $\mathcal{A}$ is a symmetric matrix. An edge denoted by the pair (j,i) represents a communication channel from j to i and $(j,i) \in \mathcal{E}$ if and only if $(i,j) \in \mathcal{E}$. The neighborhood of the ith agent is denoted by $N_i = \{j \in \mathcal{V} \mid (i,j) \in \mathcal{E}\}$. For any $i,j \in \mathcal{V}$, $a_{ij} = a_{ji} \geq 0$, and $a_{ij} > 0$ if and only if $j \in N_i$. Also, $\deg_i = \sum_{j=1}^{N} a_{ij}$ is called the degree of i, and $d^* = \max_{1\leq i\leq N} \deg_i$ is called the degree of $\mathcal{G}$. The Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \mathrm{diag}(\deg_1,\ldots,\deg_N)$. The Laplacian matrix $\mathcal{L}$ is a symmetric positive semi-definite matrix and its eigenvalues in an ascending order are denoted by $0 = \lambda_1(\mathcal{L}) \leq \lambda_2(\mathcal{L}) \leq \cdots \leq \lambda_N(\mathcal{L})$, where $\lambda_N(\mathcal{L})$ is the spectral radius of $\mathcal{L}$ and $\lambda_2(\mathcal{L})$ is called the algebraic connectivity of $\mathcal{G}$ ([34], [35]). A sequence of edges $(i_1,i_2),(i_2,i_3),\ldots(i_{k-1},i_k)$ is called a path from node i_1 to node i_k. The graph $\mathcal{G}$ is called a connected graph if for any $i,j \in \mathcal{V}$, there is a path from i to j.

B. Observer-Based Encoding-Decoding

We consider digital communication channels with limited channel capacity. At each time step, what each agent can send to its neighbors is only a coded version of its current and past measurements. Generally speaking, the encoder of the ith agent may take the following form:

$$\begin{aligned} s_i(t) = q_t(f_{i_t}(z_i(t), z_i(t-1),\ldots,z_i(0)\\ s_i(t-1),\ldots,s_i(1))),\ t=1,2,\ldots \end{aligned} \tag{2}$$

where $s_i(t)$ and $z_i(t)$ are the output and input of the encoder, respectively, f_{i_t} is a Borel measurable function and q_t is a quantizer. Note that both the structure and parameters of f_{i_t} and q_t may be time-varying and the encoder may have infinite memory. In this paper, we propose a finite memory encoder ϕ_i of agent i as

$$\begin{cases} \hat{p}_i(0) = 0\\ \hat{p}_i(t) = \hat{p}_i(t-1) + \hat{v}_i(t-1) + g(t-1)s_i(t)\\ \hat{v}_i(t) = \hat{v}_i(t-1) + g(t-1)s_i(t),\ \hat{v}_i(0) = 0\\ s_i(t) = q_t\left(\frac{z_i(t)-\hat{p}_i(t-1)-\hat{v}_i(t-1)}{g(t-1)}\right),\ t=1,2,\ldots \end{cases} \tag{3}$$

where $g(t)$ is an exponentially decaying scaling function to be defined later. In the above, $\hat{p}_i(t)$ and $\hat{v}_i(t)$ are the internal states of the encoder and $q_t(\cdot)$ is a finite-level uniform quantizer given by

$$q_t(y) = \begin{cases} 0, & \frac{-1}{2} < y < \frac{1}{2}\\ i, & \frac{2i-1}{2} \leq y < \frac{2i+1}{2}\\ & i = 1,2,\ldots,M(t)-1\\ M(t), & y \geq \frac{2M(t)-1}{2}\\ -q(-y), & y \leq \frac{-1}{2} \end{cases} \tag{4}$$

where $M(t) \in \{1,2,\ldots\}$ is the number of quantization levels of $q_t(\cdot)$.

After $s_i(t)$ is received by one of the ith agent's neighbors, say agent j, a decoder φ_{ij} will be activated:

$$\begin{cases} \hat{p}_{ij}(0) = 0\\ \hat{p}_{ij}(t) = \hat{p}_{ij}(t-1) + \hat{v}_{ij}(t-1) + g(t-1)s_i(t)\\ \hat{v}_{ij}(0) = 0\\ \hat{v}_{ij}(t) = \hat{v}_{ij}(t-1) + g(t-1)s_i(t),\ t=1,2,\ldots \end{cases} \tag{5}$$

where $\hat{p}_{ij}(t)$ and $\hat{v}_{ij}(t)$ are the outputs of the decoder.

Remark 1: In the above, $s_i(t)$ is a quantized innovation with scaling. From the dynamic (1) of the ith agent, we know that to get estimates for $p_i(t)$ and $v_i(t)$, following the standard observer design, the decoder φ_{ij} can be in the form

$$\begin{cases} \hat{p}_{ij}(t) = \hat{p}_{ij}(t-1) + \hat{v}_{ij}(t-1) + l_1 g(t-1)s_i(t)\\ \hat{v}_{ij}(t) = \hat{v}_{ij}(t-1) + u_i(t-1) + l_2 g(t-1)s_i(t) \end{cases} \tag{6}$$

where $l_1 \in \mathbb{R}$ and $l_2 \in \mathbb{R}$ are the observer gains. It can be easily verified that if $l_1 = l_2 = 1$ and the quantizer $q_t(\cdot)$ is the identity function, then (6) degenerates to the classical deadbeat posterior state observer based on output $z_i(t)$. However, since $u_i(t-1)$ is not available for the neighbors of the ith agent, we adopt decoder (5) instead.

Remark 2: From (3) and (5), we have

$$\hat{p}_{ij}(t) \equiv \hat{p}_i(t), \hat{v}_{ij}(t) \equiv \hat{v}_i(t), j \in N_i, i = 1,2,\ldots,N. \tag{7}$$

We will show that $\hat{p}_i(t)$ and $\hat{v}_i(t)$ can be viewed as the estimates for $p_i(t)$ and $v_i(t-1)$, respectively. Denote

$$\Delta_i(t) = s_i(t) - \frac{z_i(t) - \hat{p}_i(t-1) - \hat{v}_i(t-1)}{g(t-1)}$$

as the quantization error in encoder ϕ_i, $e_{p_i}(t) = \hat{p}_i(t) - p_i(t)$ as the estimation error for $p_i(t)$ and $e_{v_i}(t-1) = \hat{v}_i(t) - v_i(t-1)$ as the estimation error for $v_i(t-1)$. By (3) and some direct calculation, we get

$$\begin{cases} e_{p_i}(0) = -p_i(0)\\ e_{p_i}(t) = g(t-1)\Delta_i(t),\ t=1,2\ldots \end{cases} \tag{8}$$

and

$$\begin{cases} e_{v_i}(0) = p_i(0) + g(0)\Delta_i(1)\\ e_{v_i}(t-1) = g(t-1)\Delta_i(t) - g(t-2)\Delta_i(t-1)\\ t = 2,3\ldots. \end{cases} \tag{9}$$

It can be seen that if the quantization error $\Delta_i(t)$ is bounded, then due to the vanishing of $g(t)$, the estimation errors $e_{p_i}(t)$ and $e_{v_i}(t-1)$ will both converge to zero asymptotically as $t \to \infty$. Note that here, for the velocity estimation, there is one step delay.

Remark 3: The relationship among the estimation errors $e_{p_i}(t)$, $e_{v_i}(t-1)$ and the quantization error $\Delta_i(t)$ is not as straightforward as in the first-order case ([24]). It will be seen later that (8) and (9) will play an important role in the closed-loop analysis. Observe that the estimation errors for velocities depend on two steps of quantization errors, which, as we can see later, leads to an additional bit required for the quantizers as compared to the first-order case ([24]).

Remark 4: From the above, we can see that both the encoder (3) and the decoder (5) can be viewed as the state observers based on the output $z_i(t)$ and the quantized innovation. We call the encoder (3) an observer-based encoder and the decoder (5) an observer-based decoder. Though the velocity $v_i(t)$ is not measurable, the ith agent and its neighbors can make an estimate for the overall state $[p_i(t), v_i(t)]^T$ by using an observer-based encoder and an observer-based decoder. At each time step, each agent only needs to send the quantized innovation of its output to its neighbors, then the neighbors can use observer-based decoders to get estimates for the state of the agent. However, generally speaking, there is no separation principle for the encoder-decoder design and the control design. Compared with classical non-quantized and centralized state observers, due to the nonlinearity of the quantization and the coupling of all agents' states, the convergence of a given observer-based encoding-decoding scheme depends on the control inputs of all agents and the closed-loop dynamics of the whole network, which significantly complicates the analysis as seen below.

C. Distributed Control Law

In this paper, we aim at designing a distributed coordinated control law based on quantized communications such that

$$\lim_{t\to\infty}[p_i(t)-p_j(t)]=0,\ \lim_{t\to\infty}[v_i(t)-v_j(t)]=0. \tag{10}$$

We propose a distributed coordinated control law of the form

$$\begin{cases} u_i(0)=z_i(0) \\ u_i(t)=k_1\sum_{j\in N_i}a_{ij}[\hat{p}_{ji}(t)-\hat{p}_i(t)] \\ \qquad\quad +k_2\sum_{j\in N_i}a_{ij}[\hat{v}_{ji}(t)-\hat{v}_i(t)],\ t=1,2\ldots \end{cases} \tag{11}$$

where $k_1>0$ and $k_2>0$ are the control gains.

From (3), (5) and (11), we can see that the control input of each agent only depends on the state of its own encoder and the states of the decoders associated with the channels from its neighbors.

Remark 5: Since the states of agents are only partially measurable, the encoding-decoding scheme in [24] where agents of single integrator dynamics are considered cannot be easily extended to this case. The challenge is to design state observers and encoders-decoders jointly so that they can achieve consensus with efficient communications and computation. One natural idea is to design a state-observer for each agent and then encode and transmit the state estimate to neighbors. For example, we may adopt the following state-observer for the ith agent:

$$\begin{cases} p_{i_o}(t)=p_{i_o}(t-1)+v_{i_o}(t-1) \\ \qquad\quad +g_1(z_i(t-1)-p_{i_o}(t-1)) \\ v_{i_o}(t)=v_{i_o}(t-1)+u_i(t-1) \\ \qquad\quad +g_2(z_i(t-1)-p_{i_o}(t-1)),\ t=1,2,\ldots \end{cases} \tag{12}$$

$[p_{i_o}(t),v_{i_o}(t)]^T$ is then encoded and transmitted to the neighbors of the ith agent. However, since the control input $u_i(t-1)$ and estimation error $z_i(t-1)-p_{i_o}(t-1)$ are not available for its neighbors, to eliminate the effect of quantization and estimation errors on the final closed-loop system, we may need a more complex encoding-decoding scheme and a control law than (3), (5) and (11). Further, even if we can find such a scheme to guarantee convergence, the computation and communication loads are higher and the performance (i.e., the convergence rate under the same bit rate) is not definitely better. From the perspective of bit rate constraint and reducing computation load, we propose an integrative approach for the state-observer and encoder-decoder design.

III. Convergence Analysis

This section is devoted to the convergence analysis of the proposed distributed control law in the last section. To this end, we introduce the following notation:

$$\begin{aligned} u(t)&=[u_1(t),\ldots,u_N(t)]^T \\ \Delta(t)&=[\Delta_1(t),\ldots,\Delta_N(t)]^T \\ p(t)&=[p_1(t),\ldots,p_N(t)]^T \\ v(t)&=[v_1(t),\ldots,v_N(t)]^T \\ e_p(t)&=[e_{p_1}(t),\ldots,e_{p_N}(t)]^T \\ e_v(t)&=[e_{v_1}(t),\ldots,e_{v_N}(t)]^T \\ \delta_p(t)&=(I-J_N)p(t) \\ \delta_v(t)&=(I-J_N)v(t) \end{aligned}$$

where $J_N=\mathbf{11^T/N}$. We also define the unitary matrix

$$T_{\mathcal{L}}=\left[\frac{\mathbf{1}}{\sqrt{N}},\phi_2,\ldots,\phi_N\right] \tag{13}$$

where ϕ_i is the unit eigenvector of $\mathcal{L}$ associated with $\lambda_i(\mathcal{L})$, that is, $\phi_i^T\mathcal{L}=\lambda_i(\mathcal{L})\phi_i^T$, $\|\phi_i\|=1$, $i=2,\ldots,N$.

Under the protocol (3), (5) and (11), due to the quantization, the closed-loop system is a nonlinear discontinuous system. Generally speaking, the convergence analysis is difficult, however, by using the estimation error expressions (8) and (9), the closed-loop equation can be converted into a linear equation with time-varying disturbances, whose homogeneous part is just the closed-loop equation without quantization. Then by properly selecting the number of quantization levels, the quantizers can be kept unsaturated and the convergence of the closed-loop system can be achieved.

We make the following assumptions.

- A1) There are known positive constants C_p, C_v, $C_{\delta p}$, $C_{\delta v}$, such that $\|p(0)\|_\infty\le C_p$, $\|v(0)\|_\infty\le C_v$, $\|\delta_p(0)\|_\infty\le C_{\delta p}$, $\|\delta_v(0)\|_\infty\le C_{\delta v}$.
- A2) The communication graph $\mathcal{G}$ is connected.
- A3) $k_2>k_1$.
- A4) $\lambda_N(\mathcal{L})<(k_2-k_1)/k_2^2$.

The following lemma, whose proof can be found in Appendix, will be used in the analysis of the homogeneous part of the closed-loop system.

Lemma 3.1: Let

$$G_i(k_1,k_2)=\begin{pmatrix} 1 & 1 & 0 & 0 \\ -k_1\lambda_i(\mathcal{L}) & 1 & 0 & -k_2\lambda_i(\mathcal{L}) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{14}$$

Then,

- i) $\rho(G_i(k_1,k_2))<1$, $i=2,3,\ldots,N$ if and only if Assumptions (A2)–(A4) hold.
- ii) Let

$$\begin{cases} p(x,y)=(x+y)+1-\frac{(-2)^2}{3} \\ q(x,y)=\frac{2}{27}(-2)^3-\frac{1}{3}(-2)(x+y+1)-y. \end{cases} \tag{15}$$

If Assumptions (A3)–(A4) hold, then the eigenvalues of $G_i(k_1,k_2)$ are 0, $s_1(k_1\lambda_i,k_2\lambda_i)$, $s_2(k_1\lambda_i,k_2\lambda_i)$ and $s_3(k_1\lambda_i,k_2\lambda_i)$, where

$$\begin{cases} s_1(x,y)=\frac{2}{3}+\sqrt[3]{-\frac{q}{2}-\sqrt{(\frac{q}{2})^2+(\frac{p}{3})^3}} \\ \qquad\qquad +\sqrt[3]{-\frac{q}{2}+\sqrt{(\frac{q}{2})^2+(\frac{p}{3})^3}} \\ s_2(x,y)=\frac{2}{3}+\omega\sqrt[3]{-\frac{q}{2}-\sqrt{(\frac{q}{2})^2+(\frac{p}{3})^3}} \\ \qquad\qquad +\omega^2\sqrt[3]{-\frac{q}{2}+\sqrt{(\frac{q}{2})^2+(\frac{p}{3})^3}} \\ s_3(x,y)=\frac{2}{3}+\omega^2\sqrt[3]{-\frac{q}{2}-\sqrt{(\frac{q}{2})^2+(\frac{p}{3})^3}} \\ \qquad\qquad +\omega\sqrt[3]{-\frac{q}{2}+\sqrt{(\frac{q}{2})^2+(\frac{p}{3})^3}}. \end{cases} \tag{16}$$

In the above, the arguments x, y of $p(x,y)$ and $q(x,y)$ were omitted, and $\omega=(-1+\sqrt{3}i)/2$, where $i^2=-1$.

From Lemma 3.1, we know that if Assumptions (A2)–(A4) hold, then $G_i(k_1,k_2)$ is diagonalizable. Let $R_i(k_1,k_2)$, $i=2,3,\ldots,N$, be nonsingular matrices, such that $\|R_i(k_1,k_2)\|=1$

$$G_i(k_1,k_2)=R_i^{-1}(k_1,k_2)\widetilde{G}_i(k_1,k_2)R_i(k_1,k_2)$$

where

$$\widetilde{G}_i(k_1,k_2)=diag\{0,s_1(k_1\lambda_i(\mathcal{L}),k_2\lambda_i(\mathcal{L})) \\ s_2(k_1\lambda_i(\mathcal{L}),k_2\lambda_i(\mathcal{L})),s_3(k_1\lambda_i(\mathcal{L}),k_2\lambda_i(\mathcal{L}))\}.$$

Denote $\rho_i(k_1,k_2)=\rho(G_i(k_1,k_2))$, $\rho(k_1,k_2)=\max_{2\le i\le N}\rho_i(k_1,k_2)$, $r(k_1,k_2)=\max_{2\le i\le N}\|R_i^{-1}(k_1,k_2)\|$. In the following, the dependence of ρ_i, r and ρ on k_1 and k_2 will be omitted when there is no confusion.

The following theorem gives sufficient conditions on the control gains and network topology for the existence of finite-level quantizers to ensure the closed-loop convergence.

Theorem 3.1: Suppose Assumptions (A1)–(A4) hold. Let the scaling function $g(t)=g_0\gamma^t$, where

$$g_0\ge\frac{2(\gamma-\rho)(k_2\lambda_N(\mathcal{L})C_p+2\rho\|T_{\mathcal{L}}^{-1}\|_\infty(C_{\delta_p}+C_{\delta_v}))}{k_2\lambda_N(\mathcal{L})} \tag{17}$$

and $\gamma\in(\rho,1)$. If the numbers of quantization levels $M(t)$ of the quantizer $q_t(\cdot)$, $t=1,2,\ldots$ satisfy

$$\begin{cases} M(1)\ge\frac{1}{g_0}(C_p+C_v)-\frac{1}{2} \\ M(2)\ge\frac{1}{\gamma}-\frac{1}{2} \\ M(3)\ge M_0(\gamma,k_1,k_2)+\frac{2(k_1+k_2)d^*(C_p+C_v)}{g_0\gamma^2} \\ M(4)\ge M_0(\gamma,k_1,k_2)+\frac{k_2d^*}{\gamma^3}+\frac{2(2k_1+k_2)d^*(C_p+C_v)}{g_0\gamma^3} \end{cases} \tag{18}$$

and

$$M(t)\ge M_0(\gamma,k_1,k_2)+\frac{k_2d^*}{\gamma^3}+\lambda_N(\mathcal{L})(k_1+k_2)rd^* \\ \times\|T_{\mathcal{L}}\|_\infty\left[\frac{k_1+k_2}{\gamma^2(\gamma-\rho)}+\frac{k_2}{\gamma^3(\gamma-\rho)}\right],\ t=5,6,\ldots \tag{19}$$

where $M_0(\gamma,k_1,k_2)=1/\gamma+(1+2(k_1+k_2)d^*)/(2\gamma^2)-1/2$, then under the protocol (3), (5) and (11), the closed-loop system satisfies

$$\lim_{t\to\infty}[p_i(t)-p_j(t)]=0 \\ \lim_{t\to\infty}[v_i(t)-v_j(t)]=0,\ i,j=1,2,\ldots,N. \tag{20}$$

Furthermore, the convergence rate is given by

$$|p_i(t)-p_j(t)|=O(\gamma^t) \\ |v_i(t)-v_j(t)|=O(\gamma^t),\ i,j=1,2,\ldots,N. \tag{21}$$

Proof: The proof can be divided into three steps. First, we convert the closed-loop system into $N-1$ non-coupled linear equations with nonlinear disturbances. The disturbances are combinations of the estimation errors which are related to the quantization errors as observed from by (8) and (9). Second, we estimate the bound of the synchronization errors in terms of the quantization errors and system and control parameters. Finally, we prove the boundness of the quantization error by properly choosing the control parameters and the number of quantization levels, which will lead to the convergence of the closed-loop system.

Step 1) From (7) and (11), it follows that

$$u(t)=-k_1\mathcal{L}p(t)-k_2\mathcal{L}v(t-1) \\ -k_1\mathcal{L}e_p(t)-k_2\mathcal{L}e_v(t-1),\ t=1,2\ldots. \tag{22}$$

Substitute the control law above into the system (1), we have

$$\begin{cases} \delta_p(t+1)=\delta_p(t)+\delta_v(t) \\ \delta_v(t+1)=-k_1\mathcal{L}\delta_p(t)+\delta_v(t)-k_2\mathcal{L}\delta_v(t-1) \\ \qquad\qquad -k_1\mathcal{L}e_p(t)-k_2\mathcal{L}e_v(t-1),\ t=1,2,\ldots. \end{cases}$$

Let $\delta_p(t)=T_{\mathcal{L}}\tilde{\delta}_p(t)$, $\delta_v(t)=T_{\mathcal{L}}\tilde{\delta}_v(t)$, where $T_{\mathcal{L}}$ is defined in (13). Denote the ith components of $\tilde{\delta}_p(t)$ and $\tilde{\delta}_v(t)$ by $\tilde{\delta}_{p_i}(t)$ and $\tilde{\delta}_{v_i}(t)$, respectively. Then we have $\tilde{\delta}_{p_1}(t)=\tilde{\delta}_{v_1}(t)\equiv 0$, and

$$\begin{cases} \tilde{\delta}_{p_i}(t+1)=\tilde{\delta}_{p_i}(t)+\tilde{\delta}_{v_i}(t) \\ \tilde{\delta}_{v_i}(t+1)=\tilde{\delta}_{v_i}(t)-k_1\lambda_i(\mathcal{L})\tilde{\delta}_{p_i}(t) \\ \qquad -k_2\lambda_i(\mathcal{L})\tilde{\delta}_{v_i}(t-1)-k_1\lambda_i(\mathcal{L})\phi_i^Te_p(t) \\ \qquad -k_2\lambda_i(\mathcal{L})\phi_i^Te_v(t-1),\ t=1,2,\ldots \\ i=2,\ldots,N. \end{cases} \tag{23}$$

Denote $y_i(t)=[\tilde{\delta}_{p_i}(t),\tilde{\delta}_{v_i}(t),\tilde{\delta}_{p_i}(t-1),\tilde{\delta}_{v_i}(t-1)]^T$, then the (23) can be rewritten as

$$y_i(t+1)=G_i(k_1,k_2)y_i(t) \\ +E_i(k_1,k_2,t-1),\ t=1,2,\ldots,\ i=2,\ldots,N, \tag{24}$$

where $E_i(k_1,k_2,t-1)=[0,e_i(k_1,k_2,t-1),0,0]^T$ with $e_i(k_1,k_2,t-1)=-k_1\lambda_i(\mathcal{L})\phi_i^Te_p(t)-k_2\lambda_i(\mathcal{L})\phi_i^Te_v(t-1)$. It is clear that to get (20), we only need to prove $\lim_{t\to\infty}y_i(t)=0$, $i=2,3,\ldots,N$.

Step 2) By (24), we have

$$\|y_i(t+1)\| \le r\rho_i^t\|y_i(1)\| + r\rho_i^{t-1}\|E_i(k_1,k_2,0)\| + r\sum_{j=2}^{t}\rho_i^{t-j}\|E_i(k_1,k_2,j-1)\|,\ t=2,3\ldots. \quad (25)$$

Further, by (8) and (9), noting that $\|\phi_i\|_\infty \le \|\phi_i\| = 1$, we have

$$\begin{aligned}&\|E_i(k_1,k_2,j-1)\|\\ &= |k_1\lambda_i(\mathcal{L})\phi_i^T e_p(j) + k_2\lambda_i(\mathcal{L})\phi_i^T e_v(j-1)|\\ &\le k_1\lambda_i(\mathcal{L})\|\phi_i^T\|\|e_p(j)\|_\infty + k_2\lambda_i(\mathcal{L})\|\phi_i^T\|\|e_v(j-1)\|_\infty\\ &= (k_1+k_2)\lambda_i(\mathcal{L})g(j-1)\|\Delta(j)\|_\infty\\ &\quad + k_2\lambda_i(\mathcal{L})g(j-2)\|\Delta(j-1)\|_\infty,\ j=2,3,\ldots.\end{aligned}$$

Then it follows from (25) that

$$\begin{aligned}&\|y_i(t+1)\|\\ &\le r\rho_i^t\|y_i(1)\| + \rho_i^{t-1}\|E_i(k_1,k_2,0)\|\\ &\quad + r\sup_{1\le j\le t}\|\Delta(j)\|_\infty \frac{\rho_i^t(k_1+k_2)\lambda_i(\mathcal{L})g_0}{\gamma}\sum_{j=2}^{t}(\frac{\gamma}{\rho_i})^j\\ &\quad + r\sup_{1\le j\le t-1}\|\Delta(j)\|_\infty \frac{\rho_i^t k_2\lambda_i(\mathcal{L})g_0}{\gamma^2}\sum_{j=2}^{t}(\frac{\gamma}{\rho_i})^j\\ &\le r\rho_i^t\|y_i(1)\| + \rho_i^{t-1}\|E_i(k_1,k_2,0)\|\\ &\quad + r\sup_{1\le j\le t}\|\Delta(j)\|_\infty \frac{\rho_i^t(k_1+k_2)\lambda_i(\mathcal{L})g_0}{\gamma}\\ &\quad \times \frac{(\frac{\gamma}{\rho_i})^2[(\frac{\gamma}{\rho_i})^{t-1}-1]}{\frac{\gamma}{\rho_i}-1}\\ &\quad + r\sup_{1\le j\le t-1}\|\Delta(j)\|_\infty \frac{\rho_i^t k_2\lambda_i(\mathcal{L})g_0}{\gamma^2}\\ &\quad \times \frac{(\frac{\gamma}{\rho_i})^2[(\frac{\gamma}{\rho_i})^{t-1}-1]}{\frac{\gamma}{\rho_i}-1}\\ &\le r\rho_i^t\|y_i(1)\| + \rho_i^{t-1}\|E_i(k_1,k_2,0)\|\\ &\quad + r\sup_{1\le j\le t}\|\Delta(j)\|_\infty \frac{\gamma(\gamma^{t-1}-\rho_i^{t-1})(k_1+k_2)\lambda_i(\mathcal{L})g_0}{\gamma-\rho_i}\\ &\quad + r\sup_{1\le j\le t-1}\|\Delta(j)\|_\infty \frac{(\gamma^{t-1}-\rho_i^{t-1})k_2\lambda_i(\mathcal{L})g_0}{\gamma-\rho_i}\\ &t=2,3\ldots. \end{aligned} \quad (26)$$

By the definition of $y_i(t)$, $\tilde{\delta}_p(t)$ and $\tilde{\delta}_v(t)$, we get

$$\begin{aligned}&\|[\delta_p^T(t),\delta_v^T(t-1)]^T\|_\infty\\ &\le \max\{\|T_{\mathcal{L}}\|_\infty\|\tilde{\delta}_p(t)\|_\infty, \|T_{\mathcal{L}}\|_\infty\|\tilde{\delta}_v(t-1)\|_\infty\}\\ &= \|T_{\mathcal{L}}\|_\infty\|[\tilde{\delta}_p^T(t),\tilde{\delta}_v^T(t-1)]^T\|_\infty\\ &\le \|T_{\mathcal{L}}\|_\infty \max_{i=2,\ldots,N}\|y_i(t)\|_\infty\\ &\le \|T_{\mathcal{L}}\|_\infty \max_{i=2,\ldots,N}\|y_i(t)\|. \end{aligned} \quad (27)$$

Step 3) By Lemma A.2, we get $\sup_{t\ge1}\|\Delta(t)\|_\infty \le 1/2$. This together with (26) gives $\lim_{t\to\infty} y_i(t) \to 0$, $i=2,3,\ldots,N$, which further implies (20). Then from $\sup_{t\ge1}\|\Delta(t)\|_\infty \le 1/2$, (26) and (27), we get (21). □

Observe that the distributed control law in Theorem 3.1 relies on $T_{\mathcal{L}}$, which requires each agent to know the graph $\mathcal{L}$ and may not be practical. This restriction is relaxed by the following corollary.

Corollary 3.1: Suppose Assumptions (A1)–(A4) hold. Let the scaling function $g(t) = g_0\gamma^t$, where

$$g_0 \ge \frac{2(\gamma-\rho)(2k_2d^*C_p + 2\rho\sqrt{N}(C_{\delta_p}+C_{\delta_v}))}{k_2d*} \quad (28)$$

and $\gamma \in (\rho,1)$. If the numbers of quantization levels $M(t)$ of the quantizer $q_t(\cdot)$, $t=1,2,\ldots$ satisfy

$$\begin{cases} M(1) \ge \frac{1}{g_0}(C_p+C_v) - \frac{1}{2}\\ M(2) \ge \frac{1}{\gamma} - \frac{1}{2}\\ M(3) \ge M_0(\gamma,k_1,k_2) + \frac{2(k_1+k_2)d^*(C_p+C_v)}{g_0\gamma^2}\\ M(4) \ge M_0(\gamma,k_1,k_2) + \frac{k_2d^*}{\gamma^3}\\ \qquad + \frac{2(2k_1+k_2)d^*(C_p+C_v)}{g_0\gamma^3}\end{cases} \quad (29)$$

and

$$M(t) \ge M_0(\gamma,k_1,k_2) + \frac{k_2d^*}{\gamma^3} + 2r(k_1+k_2)(d^*)^2 \times\sqrt{N}\left[\frac{k_1+k_2}{\gamma^2(\gamma-\rho)} + \frac{k_2}{\gamma^3(\gamma-\rho)}\right],\ t=5,6,\ldots \quad (30)$$

where then under the protocol (3), (5) and (11), the closed-loop system satisfies

$$\begin{aligned}&\lim_{t\to\infty}[p_i(t)-p_j(t)] = 0\\ &\lim_{t\to\infty}[v_i(t)-v_j(t)] = 0,\ i,j=1,2,\ldots,N\end{aligned} \quad (31)$$

and the convergence rate is given by

$$\begin{aligned}&|p_i(t)-p_j(t)| = O(\gamma^t)\\ &|v_i(t)-v_j(t)| = O(\gamma^t),\ i,j=1,2,\ldots,N.\end{aligned} \quad (32)$$

Proof: Noting that $\|T_{\mathcal{L}}^{-1}\|_\infty \le \sqrt{N}\|T_{\mathcal{L}}^{-1}\|_2 = \sqrt{N}$ and $d^* \le \lambda_N \le 2d^*$, by Theorem 3.1, we get the conclusion of this corollary. □

Remark 6: From Theorem 3.1 and Corollary 3.1, we can see that the convergence factor γ can be properly chosen to tune the convergence rate of the closed-loop system. By Corollary 3.1, we may select the control parameters by the following steps. i) Choosing k_1, k_2 such that Assumptions (A3)–(A4) hold. ii) Choosing $\gamma \in (\rho,1)$ and then g_0 according to (28). iii) Choosing the number of quantization levels according to (29) and (30).

Remark 7: Corollary 3.1 tells us that to select proper g_0 and the number of quantization levels, we do not need to know $T_{\mathcal{L}}$, that is, the exact Laplacian matrix. Furthermore, Assumption A4) holds if $(k_2-k_1)/k_2 > 2d^*$, so the selection of the control gains may not need the knowledge of λ_N. However, from the definition of $\rho(k_1,k_2)$, we can see that the selection of γ needs the knowledge of the eigenvalues of the Laplacian matrix. Hence, we still need some global knowledge of the network topology to select the control parameters. In the case when the network topology can be predesigned, this is not a problem. However, in some applications, the network topology may not be known to each agent, for example, under switching topologies due to changing environment. In this situation, the problem of estimating the eigenvalues of the Laplacian matrix in a distributed manner becomes relevant. Franceschelli *et al.* ([36]) gave an algorithm to estimate the eigenvalues of a Laplacian matrix by each agent using the fast Fourier transform. The combination of the eigenvalue estimation algorithm with our proposed distributed coordinate control algorithm is an interesting future research topic.

Remark 8: From Lemma 3.1 and the proof of Theorem 3.1, we can see that A2-A4) are necessary and sufficient for the stability of the homogeneous part of the closed-loop systems (24). Since $d^* \le \lambda_N(\mathcal{L}) \le 2d^*$, we can see that a smaller degree, which implies lower local connectivity, will instead give more flexibility for selecting the control gains.

In the main theorem of [15] (Theorem 1 of [15]), the authors proved that under their algorithm, as time goes on the states of agents converge to a ball centered at the average of the initial states with radius less than or equal to the quantization interval, with probability 1. They also proved that there always exists a finite time t_c such that the states of the agents enter and stay in the ball with a positive probability when $t > t_c$. An upper bound for the mathematical expectation of the convergence time for fully connected networks and linear networks was also provided. In this paper, we focus on the case with real-valued states and the asymptotic convergence to exact synchronization. The algorithm given here can guarantee convergence to synchronization with an arbitrary precision as time goes on. In the following, we will give an analysis on the convergence time for a given precision for connected networks. For any given $\epsilon > 0$, denote $T_{p,\epsilon} = \inf\{t : \sup_{k\ge t} \max_{ij} |p_i(k) - p_j(k)| \le \epsilon\}$ and $T_{v,\epsilon} = \inf\{t : \sup_{k\ge t} \max_{ij} |v_i(k) - v_j(k)| \le \epsilon\}$, which are respectively the convergence time for the positions and velocities of all the agents with precision ϵ.

Theorem 3.2: Suppose the conditions of Theorem 3.1 hold, $\max_{ij} |p_i(t) - p_j(t)| > 0$ and $\max_{ij} |v_i(t) - v_j(t)| > 0$. Then under the protocol (3), (5) and (11), for sufficiently small $\epsilon > 0$, the convergence time for the position and velocity respectively satisfies

$$T_{p,\epsilon} \le 2 + \frac{\ln(\frac{1}{\epsilon}) + \ln C_T(k_1,k_2,\gamma)}{\ln(\frac{1}{\gamma})}$$
$$T_{v,\epsilon} \le 1 + \frac{\ln(\frac{1}{\epsilon}) + \ln C_T(k_1,k_2,\gamma)}{\ln(\frac{1}{\gamma})} \quad (33)$$

where

$$C_T(k_1,k_2,\gamma) = 2r\|T_{\mathcal{L}}\|_\infty \Bigg[2\rho\|T_{\mathcal{L}}^{-1}\|_\infty(C_{\delta_p} + C_{\delta_v}) + k_2\lambda_N(\mathcal{L})C_p + \frac{g_0(k_1+k_2)\lambda_N(\mathcal{L})}{2} + \frac{\gamma(k_1+k_2)\lambda_N(\mathcal{L})g_0 + k_2\lambda_N(\mathcal{L})g_0}{2(\gamma-\rho)}\Bigg].$$

Proof: The proof can be found in the Appendix.

Remark 9: Similar to Corollary 3.1, the constant $C_T(k_1,k_2,\gamma)$ in Theorem 3.2 can be replaced by $2r\sqrt{N}\Big[2\rho\sqrt{N}(C_{\delta_p} + C_{\delta_v}) + 2k_2d^*C_p + d^*g_0(k_1 + k_2) + (\gamma(k_1+k_2)d^*g_0 + k_2d^*g_0)/(\gamma-\rho)\Big]$, which gives us a relationship between the upper bound of the convergence time and the number of agents.

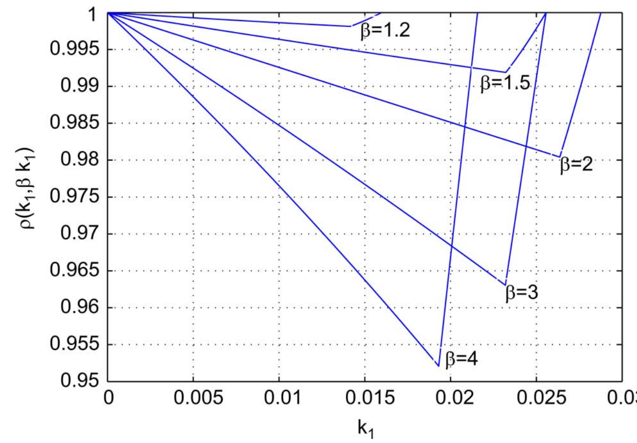

Fig. 1. Curves of $\rho(k_1, \beta k_1)$ of Example 1.

IV. Parameter Design and Performance Limit Analysis

In this section, we shall investigate controller parameter selection and analyze the asymptotic consensus convergence rate.

A. Selecting the Control Gain Ratio

Selecting the control gains k_1 and k_2 is equivalent to selecting a control gain ratio $\beta = k_2/k_1$ and the position control gain k_1. It is easily seen that Assumptions A3)-A4) hold if and only if $\beta > 1$ and $k_1 < (\beta-1)/(\beta^2\lambda_N(\mathcal{L}))$. Further $\beta = 2$ will maximize $(\beta-1)/(\beta^2\lambda_N(\mathcal{L}))$, which implies the largest stability margin of the homogeneous part of the closed-loop system (24).

1) Example 1: We consider a 10-node network with $\lambda_2(\mathcal{L}) = 1.4822$ and $\lambda_{10}(\mathcal{L}) = 8.6921$. The curves of $\rho(k_1, \beta k_1)$ with respect to k_1 with different control gain ratios β are shown in Fig. 1.

It can be seen that $\rho(k_1,\beta k_1)$ will go to 1 as $k_1 \to 0$ or $k_1 \to (\beta-1)/(\beta^2\lambda_N(\mathcal{L}))$, and $\rho(k_1,\beta k_1)$ first decreases and then increases with respect to k_1. The k_1 of the inflection point of $\rho(k_1,\beta k_1)$ reaches its maximum when $\beta = 2$. Further, it can be proved theoretically that when k_1 is sufficiently small, $\rho(k_1,\beta k_1)$ is almost a linear, monotone decreasing function of k_1. We have the following result.

Lemma 4.1: If Assumptions A2)-A4) hold, then for any given $\beta > 1$, we have

$$\rho(\mu,\beta\mu) = 1 - \frac{\beta-1}{2}\lambda_2(\mathcal{L})\mu + o(\mu),\ \mu \to 0. \quad (34)$$

Proof: The proof can be found in Appendix.

For Example 1, the curves of $\rho(k_1,\beta k_1)$ and $1-(\beta-1)\lambda_2(\mathcal{L})k_1/2$ with different β are shown in Fig. 2.

B. Selecting the Control Parameters Under a Given Communication Data Rate

In Theorem 3.1, we give a criterion for selecting the number of quantization levels (communication data rate) under given control gains and a convergence rate. In the following theorem,

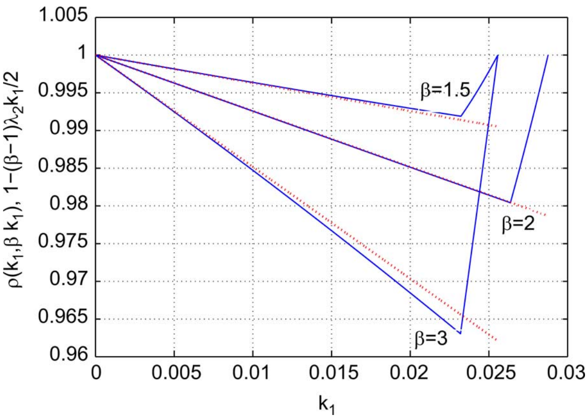

Fig. 2. Curves of $\rho(k_1, \beta k_1)$ and $1-(\beta-1)\lambda_2(\mathcal{L})k_1/2$ of Example 1 with different β, where the dot lines are for $1-(\beta-1)\lambda_2(\mathcal{L})k_1/2$ and the solid lines are for $\rho(k_1, \beta k_1)$.

we will consider how to select the control parameters under a given communication data rate.

Theorem 4.1: Suppose Assumptions A1) and A2) hold. For any given $M \geq 1$, $\beta > 1$, denote

$$\Omega_{M,\beta}=\Bigg\{(\mu,\nu)\ \Big|\ 0<\mu<\frac{\beta-1}{\beta^2\lambda_N(\mathcal{L})}, \rho(\mu,\beta\mu) < \nu < 1, \frac{1}{\nu} \leq M+\frac{1}{2}, \frac{1}{\nu} + \frac{2(1+\beta)\mu d^*+1}{2\nu^2} + \frac{\beta\mu d^*}{\nu^3} + r(\mu,\beta\mu)(1+\beta)\mu^2\lambda_N(\mathcal{L})d^*\sqrt{N}\left[\frac{1+\beta}{\nu^2(\nu-\rho(\mu,\beta\mu))} + \frac{\beta}{\nu^3(\nu-\rho(\mu,\beta\mu))}\right] \leq M+\frac{3}{2}\Bigg\}. \tag{35}$$

Then,

i) $\Omega_{M,\beta}$ is nonempty.

ii) If $(k_1,\gamma) \in \Omega_{M,\beta}$, $k_2 = \beta k_1$, and the numbers of the quantization levels of $q_t(\cdot)$ satisfy

$$\begin{cases} M(1) = M \\ M(2) = M \\ M(t) = M+1,\ t=3,4,\ldots, \end{cases} \tag{36}$$

then under the protocol given by (3), (5) and (11) with $g(t) = g_0\gamma^t$, the closed-loop system satisfies

$$\lim_{t\to\infty}[p_i(t)-p_j(t)] = 0$$
$$\lim_{t\to\infty}[v_i(t)-v_j(t)] = 0,\ i,j=1,2,\ldots,N$$

where g_0 is a constant satisfying

$$g_0 \geq \max\left\{\frac{2(2k_1+k_2)d^*(C_p+C_v)}{(M+\frac{3}{2}-\frac{1}{\gamma}-\frac{1+2(k_1+k_2)d*}{2\gamma^2}-\frac{k_2d^*}{\gamma^3})\gamma^3} \times\frac{C_p+C_v}{M+\frac{1}{2}}, (2(\gamma-\rho)(k_2\lambda_N(\mathcal{L})C_p + 2\rho\sqrt{N}(C_{\delta p}+C_{\delta v})))(k_2\lambda_N(\mathcal{L}))^{-1}\right\}. \tag{37}$$

Proof: From Lemma 4.1, we have

$$\lim_{\mu\to 0}\frac{\mu}{1-\rho(\mu,\beta\mu)} = \lim_{\mu\to 0}\frac{\mu}{1-\rho_2(\mu,\beta\mu)} = \frac{2}{(\beta-1)\lambda_2(\mathcal{L})} \tag{38}$$

which implies

$$\lim_{\mu\to 0}\lim_{\nu\to 1}\mu^2\left[\frac{1+\beta}{\nu^2(\nu-\rho(\mu,\beta\mu))}+\frac{\beta}{\nu^3(\nu-\rho(\mu,\beta\mu))}\right] = 0, \forall\,\beta>1.$$

From the aforementioned, noting that the $\lim_{\mu\to 0} r(\mu,\beta\mu)$ exists, and (35), we have (i).

For any given integer $M \geq 1$ and constant $\beta > 1$, if $(k_1,\gamma) \in \Omega_{M,\beta}$, $k_2 = \beta k_1$, (36) and (37) hold, then it is easily verified that $\gamma \in (\rho(k_1,k_2),1)$, Assumptions A3)-A4) and (18) hold. Then noting that $\|T_{\mathcal{L}}\|_\infty \leq \sqrt{N}\|T_{\mathcal{L}}\| = \sqrt{N}$ and $\|T_{\mathcal{L}}^T\|_\infty \leq \sqrt{N}\|T_{\mathcal{L}}^T\| = \sqrt{N}$, we know that (17) and (19) also hold. By Theorem 3.1, we get ii). □

Remark 10: In [24], it is shown that for a connected network with first-order agents, average-consensus can be achieved with an exponential convergence rate based on merely 1-bit information exchange between agents. Here, we prove that for the case with second-order agents, 2-bit quantizers suffice for the exponential asymptotic synchronization of agents' states. Compared with [24], from (A.2), we can see that the additional bit is used to overcome the uncertainty in estimating the velocity of the agent.

Remark 11: Compared with [24], the performance limit analysis for the second order agents with partial measurable states is much more challenging. In [24], the spectral radius of the closed-loop matrix has the simple form: $\max_{2\leq i\leq N}|1-h\lambda_i(\mathcal{L})|$, where h is the control gain. In this paper, it is very difficult to get an explicit expression for the relationship between the closed-loop spectral radius $\rho(k_1,k_2)$ and the eigenvalues of the Laplacian matrix. By differential mean theorem and limit analysis, we develop Lemma 4.1 to give a linear approximation of $\rho(k_1,k_2)$ with respect to the control gains and the algebraic connectivity. From (38), we can see that Lemma 4.1 plays a vital role in establishing Theorem 4.1. Different from [24], there is also no explicit relationship between the stability margin $1-\rho(k_1,\beta k_1)$ and the control gain k_1, which also poses a significant challenge in the asymptotic convergence rate analysis as seen later in Section IV-C.

1) Example 2: We consider a network with 10 nodes and $0-1$ weights, which means that $a_{ij} = 1$, if $(i,j) \in \mathcal{E}$, otherwise, $a_{ij} = 0$. The edges of the graph are randomly generated according to $P(i,j) \in \mathcal{E} = 0.9$, for any unordered pair (i,j). Here, $\lambda_2(\mathcal{L}) = 7$, $\lambda_{10}(\mathcal{L}) = 10.0000$. The initial states are chosen as $p_i(0) = i$ and $v_i(0) = i/10$, $i=1,\ldots,10$. The control gain $k_1 = 0.0009$ and $k_2 = 0.0018 (\beta = 2)$, which give $\rho(0.0009, 0.0018) = 0.9968$. The scaling factor γ is taken as 0.9998. According to Theorem 3.1, the 2-bit quantizer can be used. The evolution of the states is shown in Fig. 3. It can be seen that both the positions and the velocities of the agents are asymptotically synchronized. Next, we set $\gamma = 0.9975$. In this

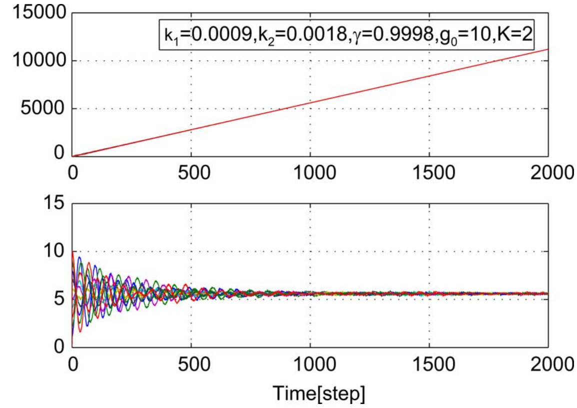

Fig. 3. Trajectories of states of Example 2 with $\gamma = 0.9998$ and $M = 2$.

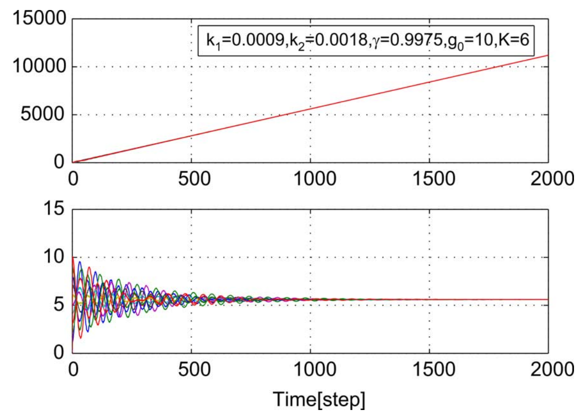

Fig. 4. Trajectories of states of Example 2 $\gamma = 0.9975$ and $M = 6$.

case, the number of quantization levels is required to be at least 6, so we take $M = 6$. The evolution of the states is shown in Fig. 4. We can see that the convergence becomes faster.

C. Asymptotic Convergence Rate

From Theorem 4.1, we can see that, for any given integer $M \geq 2$, the control gains k_1, k_2 and the convergence factor γ can be selected properly to ensure convergence of the closed-loop system under the communication data rate of $\lceil \log_2(2M) \rceil$ bits per step between agents. Since smaller γ will lead to faster convergence, an interesting question is what is the infimum of γ we can achieve under a given communication data rate. In the following, we will answer this question for large scale networks.

Theorem 4.2: Suppose Assumption A2) holds. For any given integer $M \geq 1$ and constant $\beta > 1$, let $\Omega_{M,\beta}$ be defined in Theorem 4.1. Then

$$\lim_{N\to\infty} \frac{1 - \inf_{\beta>1} \inf_{(k_1,\gamma)\in\Omega_{M,\beta}} \gamma}{1 - \exp\{-\frac{MQ_N^2}{16r_0\sqrt{N}}\}} = 1 \tag{39}$$

where $Q_N = \lambda_2(\mathcal{L})/\lambda_N(\mathcal{L})$, $r_0 = \lim_{\mu\to 0} r(\mu, \beta\mu)$ is a positive constant independent of β.

Proof: The equality (39) is a direct corollary of the following lemmas.

Lemma 4.2: Suppose Assumption A2) holds. For any given integer $M \geq 1$ and constant $\beta > 1$, let $\Omega_{M,\beta}$ be defined in Theorem 4.1. Then

$$\limsup_{N\to\infty} \frac{1 - \inf_{\beta>1} \inf_{(k_1,\gamma)\in\Omega_{M,\beta}} \gamma}{1 - \exp\{-\frac{MQ_N^2}{16r_0\sqrt{N}}\}} \leq 1. \tag{40}$$

Lemma 4.3: Suppose Assumption A2) holds. For any given integer $M \geq 1$, and constant $\beta > 1$, we have

$$\liminf_{N\to\infty} \frac{1 - \inf_{\beta>1} \inf_{(k_1,\gamma)\in\Omega_{M,\beta}} \gamma}{1 - \exp\{-\frac{MQ_N^2}{16r_0\sqrt{N}}\}} \geq 1. \tag{41}$$

The proofs of Lemmas 4.2 and 4.3 can be found in Appendix.

Remark 12: Theorem 4.2 tells us that, for a given communication data rate: $\lceil \log_2(2(M+1)) \rceil$ bits per step, as the number of agents increases to infinity, the highest asymptotic convergence rate which we can achieve is $O(\exp - MQ_N^2 t/\sqrt{N})$. It can be seen that the asymptotic convergence rate is closely related to the communication data rate, the network synchronizability Q_N and the scale of the network, which verifies again that the ratio of algebraic connectivity to the spectral radius of graph Laplacian is an important factor that determines the performance limit of the whole network.

Remark 13: Carli *et al.* proposed dynamic encoding-decoding schemes for asymptotic average-consensus with uniform and logarithmic quantizers respectively ([20]). Compared with [20], the innovation of our paper is summarized as follows. i) For dynamic uniform quantizers, [20] did not give the convergence analysis of their proposed algorithm. For dynamic logarithmic quantizers, [20] considered the infinite-level case. In this paper, we consider dynamic finite-level uniform quantizers and give a rigorous convergence analysis for our protocol. We derive sufficient conditions on the network topology, the control gains and communication data-rate to ensure asymptotic synchronization, and prove that for a connected network, 2-bit quantizers can guarantee the exponential convergence of the closed-loop system. Our conditions on the network topology and the control gains are also necessary for the stability of the closed-loop matrix when there is no quantization. ii) We give a rigorous performance analysis. It is shown that the convergence rate of our control protocol is exponentially fast and can be designed by selecting the convergence factor of the scaling function. We also discuss the relationship between the performance limit and the parameters of the network and system, which link the highest asymptotic convergence rate to the communication data rate, the network topology and the scale of the network. iii) [20] considered the first-order system, and in this paper, we consider second-order dynamics with partially measurable states and propose an integrative approach for observer and encoder-decoder co-design.

V. Concluding Remarks

In this paper, distributed coordination of discrete-time second-order multi-agent systems over finite bandwidth digital communication networks has been considered. The position of each agent is measurable, while the velocity is not. A quantized-observer based encoding-decoding scheme was proposed, which integrates the state observation with coding/decoding. A distributed coordinated control law was given in terms of the states of the encoders and decoders. It was proved that, for a connected network, by properly selecting the control gains both for the position and velocity, 2-bit quantizers can ensure the asymptotic synchronization of the states of agents. The control parameter selection and performance of the closed-loop system were also discussed. It was shown that the second-smallest and largest eigenvalues of the Laplacian matrix play important roles in the closed-loop performance and the highest asymptotic convergence rate can be described as a function of the number of quantization levels, the number of agents and the ratio of the second-smallest to the largest eigenvalues of the Laplacian matrix.

In this paper, as a preliminary research, we assume that the communication channels are noiseless and delay free. The case with noisy channels, link failures and time-delay is an interesting topic for future research. In some applications, the dynamics of agents can be described by high-order linear models, how to design an encoding-decoding scheme and a coordinated control law for general linear systems remains challenging.

Appendix

Lemma A.1: Let λ, k_1 and k_2 be positive constants. Then the roots of polynomial $s^3 - 2s^2 + (k_1\lambda + k_2\lambda + 1)s - k_2\lambda$ are all inside the unit circle if and only if i) $k_2 > k_1$ and ii) $\lambda < k_2 - k_1/k_2^2$. Furthermore, if i) and ii) hold, then the polynomial has a real nonzero root and one pair of conjugate complex roots.

Proof: By Jury criteria, the roots of $s^3 - 2s^2 + (k_1\lambda + k_2\lambda + 1)s - k_2\lambda$ are all inside the unit circle if and only if

$$\begin{cases} k_2\lambda < 1 \\ 1 - k_2^2\lambda^2 > 1 - (k_2 - k_1)\lambda \end{cases}$$

i.e., $k_2 > k_1$ and $\lambda < (k_2 - k_1)/k_2^2$.

Let $p(k_1\lambda, k_2\lambda)$ and $q(k_1\lambda, k_2\lambda)$ be given by (15) evaluated at $(k_1\lambda, k_2\lambda)$, then the roots of $s^3 - 2s^2 + (k_1\lambda + k_2\lambda + 1)s - k_2\lambda$ are given by (16) evaluated at $(k_1\lambda, k_2\lambda)$.

In the following, we prove that if i) and ii) hold, then $(q/2)^2 + (p/3)^3 > 0$. Denote $\beta = k_2/k_1$, $k_\lambda = k_1\lambda$ and

$$D_\beta(k_\lambda) = \frac{1}{27}\left((1+\beta)k_\lambda - \frac{1}{3}\right)^3 + \frac{1}{4}\left[\frac{2}{27} + (\frac{2}{3} - \frac{1}{3}\beta)k_\lambda\right]^2.$$

Then we have $(q/2)^2 + (p/3)^3 = D_\beta(k_\lambda)$ and

$$\frac{dD_\beta(k_\lambda)}{dk_\lambda} = \frac{1}{9}(1+\beta)^3 k_\lambda^2 + \left[\frac{4}{27} + \frac{10}{27}\beta - \frac{1}{54}\beta^2\right]k_\lambda + \frac{1}{27}.$$

Denote $\overline{D}(\beta) = \left[4/27 + (10/27)\beta - (1/54)\beta^2\right]^2 - (4/243)(1+\beta)^3$. Next, we prove that

$$\frac{dD_\beta(k_\lambda)}{dk_\lambda} > 0,\ \forall\, k_\lambda \in [0, k_\lambda < \frac{\beta - 1}{\beta^2}),\ \forall\, \beta > 1. \quad \text{(A.1)}$$

We consider this according to three situations.

I) $\overline{D}(\beta) < 0$, $\beta > 1$. In this case, we always have $dD_\beta(k_\lambda)/dk_\lambda > 0, \forall\, k_\lambda \in [0, \infty)$.

II) $\overline{D}(\beta) \geq 0$, $4/27 + (10/27)\beta - (1/54)\beta^2 > 0$, $\beta > 1$. In this case, it can be seen that the roots (or root) of $dD_\beta(k_\lambda)/dk_\lambda = 0$ are (is) negative, then from $dD_\beta(k_\lambda)/dk_\lambda\Big|_{k_\lambda = 0} = 1/27$, we can see that $dD_\beta(k_\lambda)/dk_\lambda > 0, \forall\, k_\lambda \in [0, \infty)$.

III) $\overline{D}(\beta) \geq 0, 4/27 + (10/27)\beta - (1/54)\beta^2 < 0, \beta > 1$. In this case, it can be verified that

$$\frac{\beta - 1}{\beta^2} \leq \frac{-(\frac{4}{27} + \frac{10}{27}\beta - \frac{1}{54}\beta^2) - \sqrt{\overline{D}(\beta)}}{\frac{2}{9}(1+\beta)^3}.$$

Then we have $dD_\beta(k_\lambda)/dk_\lambda > 0, \forall\, k_\lambda \in [0, k_\lambda < (\beta - 1)/\beta^2)$. Thus we can conclude that (A.1) holds. Then noting that $D_\beta(0) = 0$, we get $(q/2)^2 + (p/3)^3 > 0$.

Since $(q/2)^2 + (p/3)^3 > 0$, from (16), we can see that s_1 is real and s_2 and s_3 are conjugate complex. Then by i), it is easily verified that $s_1 \neq 0$. □

Proof of Lemma 3.1: The characteristic equation of $G_i(k_1, k_2)$, $i = 2, 3, \ldots, N$, is given by

$$s(s^3 - 2s^2 + (k_1\lambda_i(\mathcal{L}) + k_2\lambda_i(\mathcal{L}) + 1)s - k_2\lambda_i(\mathcal{L})) = 0.$$

By Assumption A2), we know that $\lambda_i(\mathcal{L}) > 0, i = 2, 3, \ldots, N$. Then by Assumptions A3)-A4) and Lemma A.1, we get the sufficiency of i) and ii). The necessity of Assumption A2) in i) can be easily verified by letting $\lambda_2(\mathcal{L}) = 0$. Then by Lemma A.1, the the necessity of Assumptions A3)-A4) is easily verified. □

Lemma A.2: Suppose the conditions of Theorem 3.1 hold. Then under the protocol (3), (5) and (11), the closed-loop system satisfies $\sup_{t\geq 1} \|\Delta(t)\|_\infty \leq 1/2$.

Proof: From (3), (18) and A1), we have $\|\Delta(1)\|_\infty \leq 1/2$. From (1), (8), (9) and (11), we have

$$\begin{aligned} s_i(2) &= q\left(\frac{p_i(2) - \hat{p}_i(1) - \hat{v}_i(1)}{g(1)}\right) \\ &= q\left(\frac{p_i(1) - \hat{p}_i(1) + v_i(0) - \hat{v}_i(1) + u_i(0)}{g(1)}\right) \\ &= q\left(\frac{u_i(0) - p_i(0) - 2g(0)\Delta_i(1)}{g(1)}\right) \\ &= q\left(\frac{-2\Delta_i(1)}{\gamma}\right)\ i = 1, 2, \ldots, N. \end{aligned}$$

Then by (18) and A1), noting that $\|\Delta(1)\|_\infty \leq 1/2$, we get $\|\Delta(2)\|_\infty \leq 1/2$. From (1), (8), (9) and (22), we know that

$$\begin{aligned} &\frac{p(t+1) - \hat{p}(t) - \hat{v}(t)}{g(t)} \\ &= \frac{p(t) - \hat{p}(t) + v(t-1) - \hat{v}(t) + u(t-1)}{g(t)} \\ &= \frac{-2g(t-1)\Delta(t) + g(t-2)\Delta(t-1) + u(t-1)}{g(t)} \\ &= \frac{-k_1\mathcal{L}p(t-1) - k_2\mathcal{L}v(t-2)}{g(t)} \\ &\quad + \frac{-k_1\mathcal{L}e_p(t-1) - k_2\mathcal{L}e_v(t-2)}{g(t)} \\ &\quad + \frac{-2\Delta(t)}{\gamma} + \frac{\Delta(t-1)}{\gamma^2},\ t = 2, 3, \ldots, \end{aligned} \quad \text{(A.2)}$$

which implies that

$$\begin{aligned}\frac{p(3)-\hat{p}(2)-\hat{v}(2)}{g(2)} &= \frac{-2\Delta(2)}{\gamma}+\frac{\Delta(1)}{\gamma^2}\\ &\quad+\frac{-(k_1+k_2)\mathcal{L}(v(0)+p(0))}{g(2)}\\ &\quad+\frac{-g(0)(k_1+k_2)\mathcal{L}\Delta(1)}{g(2)}\end{aligned}\tag{A.3}$$

and

$$\begin{aligned}&\frac{p(4)-\hat{p}(3)-\hat{v}(3)}{g(3)}\\ &\quad=\frac{-2\Delta(3)}{\gamma}+\frac{\Delta(2)}{\gamma^2}\\ &\qquad+\frac{-k_1\mathcal{L}(p(0)+2v(0)+u(0))-k_2\mathcal{L}(v(0)+u(0))}{g(3)}\\ &\qquad+\frac{k_2g(0)\mathcal{L}\Delta(1)-(k_1+k_2)\mathcal{L}g(1)\Delta(2)}{g(3)}.\end{aligned}\tag{A.4}$$

From (A.3), (18), and A1), noting that $\|\Delta(1)\|_\infty \le 1/2$ and $\|\Delta(2)\|_\infty \le 1/2$, we have $\|\Delta(3)\|_\infty \le 1/2$. Similarly, by (A.4), (18) and A1), we have $\|\Delta(4)\|_\infty \le 1/2$.

From (8) and (9), we have

$$\begin{aligned}&\max_{2\le i\le N}\|E_i(k_1,k_2,0)\|\\ &\quad=\max_{2\le i\le N}|k_2\lambda_i\phi_i^Tp(0)+g_0(k_1+k_2)\lambda_i\phi_i^T\Delta(1)|\\ &\quad\le k_2\lambda_N\|p(0)\|_\infty+\frac{g_0(k_1+k_2)\lambda_N}{2}.\end{aligned}\tag{A.5}$$

From the definition of $y_i(t)$, noting that $v(1)=v(0)+u(0)=v(0)+p(0)$, we have

$$\begin{aligned}&\max_{2\le i\le N}\|y_i(1)\|\\ &\quad\le 2\max_{2\le i\le N}\|y_i(1)\|_\infty\\ &\quad=2\max\{\|\tilde{\delta}_p(1)\|_\infty,\|\tilde{\delta}_v(1)\|_\infty,\|\tilde{\delta}_p(0)\|_\infty,\|\tilde{\delta}_v(0)\|_\infty\}\\ &\quad\le 2\|T_{\mathcal{L}}^{-1}\|_\infty\max\{\|\delta_p(1)\|_\infty,\|\delta_v(1)\|_\infty\\ &\qquad\times\|\delta_p(0)\|_\infty,\|\delta_v(0)\|_\infty\}\\ &\quad\le 2\|T_{\mathcal{L}}^{-1}\|_\infty(\|\delta_p(0)\|_\infty+\|\delta_v(0)\|_\infty).\end{aligned}\tag{A.6}$$

Suppose that for any given integer $t\ge 4$, $\|\Delta(t)\|_\infty\le 1/2$, then at time $t+1$, from (17), (26), (27), (A.2), (A.5), and (A.6), we have

$$\begin{aligned}&\left\|\frac{p(t+1)-\hat{p}(t)-\hat{v}(t)}{g(t)}\right\|_\infty\\ &\quad\le\frac{2\|\Delta(t)\|_\infty}{\gamma}+\frac{\|\Delta(t-1)\|_\infty}{\gamma^2}\\ &\qquad+\frac{k_2\|\mathcal{L}\|_\infty\|\Delta(t-2)\|_\infty}{\gamma^3}\\ &\qquad+\frac{(k_1+k_2)\|\mathcal{L}\|_\infty\|\Delta(t-1)\|_\infty}{\gamma^2}\\ &\qquad+\frac{k_1\|\mathcal{L}\|_\infty\|\delta_p(t-1)\|_\infty+k_2\|\mathcal{L}\|_\infty\|\delta_v(t-2)\|_\infty}{g(t)}\\ &\quad\le\frac{2\|\Delta(t)\|_\infty}{\gamma}+\frac{\|\Delta(t-1)\|_\infty}{\gamma^2}\\ &\qquad+\frac{2(k_1+k_2)d^*\|\Delta(t-1)\|_\infty}{\gamma^2}+\frac{2k_2d^*\|\Delta(t-2)\|_\infty}{\gamma^3}\\ &\qquad+\frac{2(k_1+k_2)d^*\|T_{\mathcal{L}}\|_\infty\max_{2\le i\le N}\|y_i(t-1)\|}{g(t)}\\ &\quad\le\frac{2\|\Delta(t)\|_\infty}{\gamma}+\frac{(2(k_1+k_2)d^*+1)\|\Delta(t-1)\|_\infty}{\gamma^2}\\ &\qquad+\frac{2k_2d^*\|\Delta(t-2)\|_\infty}{\gamma^3}+2r(k_1+k_2)d^*\|T_{\mathcal{L}}\|_\infty\\ &\qquad\times\max_{2\le i\le N}\left[\frac{\rho_i^{t-2}}{g_0\gamma^t}\|y_i(1)\|+\frac{\rho_i^{t-3}}{g_0\gamma^t}\|E_i(k_1,k_2,0)\|\right.\\ &\qquad+\sup_{1\le j\le t-2}\|\Delta(j)\|_\infty\frac{(k_1+k_2)\lambda_N(1-(\frac{\rho_i}{\gamma})^{t-3})}{\gamma^2(\gamma-\rho_i)}\\ &\qquad\left.+\sup_{1\le j\le t-3}\|\Delta(j)\|_\infty\frac{k_2\lambda_N(1-(\frac{\rho_i}{\gamma})^{t-3})}{\gamma^3(\gamma-\rho_i)}\right]\\ &\quad\le\frac{1}{\gamma}+\frac{2(k_1+k_2)d^*+1}{2\gamma^2}+\frac{k_2d^*}{\gamma^3}+2r(k_1+k_2)d^*\|T_{\mathcal{L}}\|_\infty\\ &\qquad\times\max_{2\le i\le N}\left\{(\frac{\rho_i}{\gamma})^{t-3}\Big[\frac{k_2\lambda_N\|p(0)\|_\infty}{g_0\gamma^3}+\frac{(k_1+k_2)\lambda_N}{2\gamma^3}\right.\\ &\qquad+\frac{\rho_i}{g_0\gamma^3}\max_{2\le i\le N}\|y_i(1)\|\Big]\\ &\qquad\left.+\Big[\frac{(k_1+k_2)\lambda_N}{2\gamma^2(\gamma-\rho_i)}+\frac{k_2\lambda_N}{2\gamma^3(\gamma-\rho_i)}\Big]\Big(1-(\frac{\rho_i}{\gamma})^{t-3}\Big)\right\}\\ &\quad\le\frac{1}{\gamma}+\frac{2(k_1+k_2)d^*+1}{2\gamma^2}+\frac{k_2d^*}{\gamma^3}+2r(k_1+k_2)d^*\|T_{\mathcal{L}}\|_\infty\\ &\qquad\times\max\left\{\frac{k_2\lambda_N\|p(0)\|_\infty}{g_0\gamma^3}+\frac{(k_1+k_2)\lambda_N}{2\gamma^3}\right.\\ &\qquad+\frac{2\rho}{g_0\gamma^3}\|T_{\mathcal{L}}^{-1}\|_\infty(\|\delta_v(0)\|_\infty+\|\delta_p(0)\|_\infty)\\ &\qquad\left.\frac{(k_1+k_2)\lambda_N}{2\gamma^2(\gamma-\rho)}+\frac{k_2\lambda_N}{2\gamma^3(\gamma-\rho)}\right\}\\ &\quad\le\frac{1}{\gamma}+\frac{2(k_1+k_2)d^*+1}{2\gamma^2}+\frac{k_2d^*}{\gamma^3}\\ &\qquad+r(k_1+k_2)\lambda_Nd^*\|T_{\mathcal{L}}\|_\infty\left(\frac{k_1+k_2}{\gamma^2(\gamma-\rho)}+\frac{k_2}{\gamma^3(\gamma-\rho)}\right).\end{aligned}$$

From the above and (19), we have $\|\Delta(t+1)\|_\infty\le 1/2$. Then by induction, we know that $\sup_{t\ge 1}\|\Delta(t)\|_\infty\le 1/2$. □

Proof of Theorem 3.2: From Theorem 3.1, (26), (A.5) and (A.6), we have

$$\begin{aligned}&\max_{i=2,\ldots,N}\|y_i(t)\|\\ &\quad\le 2r\rho^{t-1}\|T_{\mathcal{L}}^{-1}\|_\infty(C_{\delta_p}+C_{\delta_v})\\ &\qquad+r\rho^{t-2}(k_2\lambda_N(\mathcal{L})C_p+\frac{g_0(k_1+k_2)\lambda_N(\mathcal{L})}{2})\\ &\qquad+r\frac{\gamma^{t-2}[\gamma(k_1+k_2)\lambda_Ng_0+k_2\lambda_N(\mathcal{L})g_0]}{2(\gamma-\rho)}\\ &\quad\le\gamma^{t-2}[2\rho\|T_{\mathcal{L}}^{-1}\|_\infty(C_{\delta_p}+C_{\delta_v})\\ &\qquad+(k_2\lambda_N(\mathcal{L})C_p+\frac{g_0(k_1+k_2)\lambda_N(\mathcal{L})}{2})\\ &\qquad+\frac{\gamma(k_1+k_2)\lambda_N(\mathcal{L})g_0+k_2\lambda_Ng_0}{2(\gamma-\rho)}]r,\ t=3,4\ldots\end{aligned}$$

which together with (27) leads to

$$\begin{aligned}\max_{ij}|p_i(t)-p_j(t)| &\le 2\|\delta_p(t)\|_\infty\\ &\le 2\|T_{\mathcal{L}}\|_\infty \max_{i=2,\ldots,N}\|y_i(t)\|\\ &\le \gamma^{t-2}C_T(k_1,k_2,\gamma),\ t=3,4,\ldots.\end{aligned} \quad \text{(A.7)}$$

Similarly, we have

$$\max_{ij}|v_i(t)-v_j(t)| \le \gamma^{t-1}C_T(k_1,k_2,\gamma),\ t=2,3\ldots. \quad \text{(A.8)}$$

Note that for sufficiently small $\epsilon > 0$, we have $\ln(1/\epsilon)+\ln C_T(k_1,k_2)/\ln(1/\gamma)\ge 0$. This together with (A.7) and (A.8) leads to (33). □

Proof of Lemma 4.1: From (16), we get $\lim_{\mu\to 0}s_1(\mu\lambda,\beta\mu\lambda)=0$ and $\lim_{\mu\to 0}|s_2(\mu\lambda,\beta\mu\lambda)|=\lim_{\mu\to 0}|s_3(\mu\lambda,\beta\mu\lambda)|=1,\ \forall\ \lambda>0$. So there exists $\kappa_1\in(0,(\beta-1)/(\beta^2\lambda_N)$, such that

$$\rho_i(\mu,\beta\mu)=|s_2(\mu\lambda_i,\beta\mu\lambda_i)|,\ \forall\,\mu\in(0,\kappa_1]\quad i=2,\ldots,N. \quad \text{(A.9)}$$

Denote $x_\beta(\alpha)=(-q(\alpha,\beta\alpha)/2-((q(\alpha,\beta\alpha)/2)^2+(p(\alpha,\beta\alpha)/3)^3)^{1/2})^{1/3}$, $y_\beta(\alpha)=(-q(\alpha,\beta\alpha)/2+((q(\alpha,\beta\alpha)/2)^2+(p(\alpha,\beta\alpha)/3)^3)^{1/2})^{1/3}$. Then by (A.9), we have

$$\rho_i(\mu,\beta\mu)=\left(\left(\frac{2}{3}-\frac{1}{2}(x_\beta(\mu\lambda_i)+y_\beta(\mu\lambda_i))\right)^2+\frac{3}{4}(x_\beta(\mu\lambda_i)-y_\beta(\mu\lambda_i))^2\right)^{1/2}\quad \forall\,\mu\in(0,\kappa_1]. \quad \text{(A.10)}$$

Denote

$$\rho_\beta(\alpha)=\left(\left(\frac{2}{3}-\frac{1}{2}(x_\beta(\alpha)+y_\beta(\alpha))\right)^2+\frac{3}{4}(x_\beta(\alpha)-y_\beta(\alpha))^2\right)^{1/2},\ \alpha>0. \quad \text{(A.11)}$$

Now we prove that

$$\lim_{a\to 0^+}\frac{d\rho_\beta(\alpha)}{d\alpha}=\frac{1-\beta}{2}. \quad \text{(A.12)}$$

By some direct calculation, we have

$$\begin{aligned}\frac{d\rho_\beta(\alpha)}{d\alpha} &=\frac{1}{2}\left(\left(\frac{2}{3}-\frac{1}{2}(x_\beta(\alpha)+y_\beta(\alpha))\right)^2+\frac{3}{4}(x_\beta(\alpha)-y_\beta(\alpha))^2\right)^{-1/2}\\ &\quad\times\left\{-\left(\frac{2}{3}-\frac{1}{2}(x_\beta(\alpha)+y_\beta(\alpha)\right)(x'_\beta(\alpha)+y'_\beta(\alpha))\right.\\ &\quad\left.+\frac{3}{2}(x_\beta(\alpha)-y_\beta(\alpha))(x'_\beta(\alpha)-y'_\beta(\alpha))\right\}\end{aligned} \quad \text{(A.13)}$$

where

$$\begin{aligned}x'_\beta(\alpha)&=\frac{1}{3}\left[-\frac{q(\alpha)}{2}-\sqrt{\frac{q^2(\alpha)}{4}+\frac{p^3(\alpha)}{27}}\right]^{-2/3}\\ &\quad\times\left[-\frac{q'(\alpha)}{2}-\frac{\frac{q'(\alpha)q(\alpha)}{2}+\frac{p^2(\alpha)p'(\alpha)}{9}}{2\sqrt{\frac{q^2(\alpha)}{4}+\frac{p^3(\alpha)}{27}}}\right].\\ y'_\beta(\alpha)&=\frac{1}{3}\left[-\frac{q(\alpha)}{2}+\sqrt{\frac{q^2(\alpha)}{4}+\frac{p^3(\alpha)}{27}}\right]^{-2/3}\\ &\quad\times\left[-\frac{q'(\alpha)}{2}+\frac{\frac{q'(\alpha)q(\alpha)}{2}+\frac{p^2(\alpha)p'(\alpha)}{9}}{2\sqrt{\frac{q^2(\alpha)}{4}+\frac{p^3(\alpha)}{27}}}\right].\end{aligned}$$

From the above, we have

$$\lim_{\alpha\to 0}(x'_\beta(\alpha)+y'_\beta(\alpha))=54\left[\frac{1}{27}\left(\frac{2}{3}-\frac{\beta}{3}\right)+\frac{1}{81}(1+\beta)\right]+\beta-2 \quad \text{(A.14)}$$

and similarly,

$$\lim_{\alpha\to 0}(x_\beta(\alpha)-y_\beta(\alpha))(x'_\beta(\alpha)-y'_\beta(\alpha))=18\left[\frac{1}{27}\left(\frac{2}{3}-\frac{\beta}{3}\right)+\frac{1}{81}(1+\beta)\right]. \quad \text{(A.15)}$$

Combining (A.13), (A.14) and (A.15), we have (A.12).

From (A.10), (A.11) and (A.12), noting that $\beta>1$, we know that there exists $\kappa_2\in(0,\kappa_1]$, such that

$$\rho(\mu,\beta\mu)=\rho_2(\mu,\beta\mu),\ \forall\,\mu\in(0,\kappa_2]. \quad \text{(A.16)}$$

From (A.10), noting that $\rho_i(\mu,\beta\mu)$ is continuously differentiable with respect to μ on $(0,\kappa_2]$, then by the differential mean value theorem, we get

$$\begin{aligned}&\rho_2(\mu,\beta\mu)\\ &=1+\mu\left.\frac{d\rho_2(x,\beta x)}{dx}\right|_{x=\xi}=1-\frac{\mu(\beta-1)\lambda_2(\mathcal{L})}{2}\\ &\quad+\mu\left(\left.\frac{d\rho_\beta(\alpha)}{d\alpha}\right|_{\alpha=\lambda_2\xi}\lambda_2(\mathcal{L})+\frac{(\beta-1)\lambda_2(\mathcal{L})}{2}\right)\\ &\exists\xi\in(0,\mu),\ \forall\,\mu\in(0,\kappa_2].\end{aligned}$$

This together with (A.12) and (A.16) leads to (34). □

In the proof of Lemmas 4.2–4.3, for abbreviation, we denote $\lambda_2(\mathcal{L})$ and $\lambda_N(\mathcal{L})$ by λ_2 and λ_N, respectively .

Proof of Lemma 4.2: For any given integer $M\ge 1$, constants $\epsilon_0\in(0,1)$ and $\beta>1$, denote

$$\begin{aligned}\Omega_{M,\beta,\epsilon_0}=\Bigg\{(\mu,\nu)\,\Bigg|\,&0<\mu<\frac{\beta-1}{\beta^2\lambda_N}\\ &\nu=1-\epsilon_0(1-\rho(\mu,\beta\mu))\\ &\frac{1}{\nu}+\frac{2(1+\beta)\mu d^*+1}{2\nu^2}+\frac{\beta\mu d^*}{\nu^3}\\ &+r(\mu,\beta\mu)(1+\beta)\mu^2\lambda_N d^*\sqrt{N}\end{aligned}$$

$$\times\left[\frac{1+\beta}{\nu^2(\nu-\rho(\mu,\beta\mu))}+\frac{\beta}{\nu^3(\nu-\rho(\mu,\beta\mu))}\right]$$
$$\leq M+\frac{3}{2}\bigg\}.$$

First, we prove that there exists an integer $N_0(\beta, M)$, such that

$$\Omega_{M,\beta}=\bigcup_{\epsilon_0\in(0,1)}\Omega_{M,\beta,\epsilon_0},\ \forall\, N\geq N_0. \tag{A.17}$$

For any given $\epsilon_0\in(0,1)$, from the definition of $\Omega_{M,\beta,\epsilon_0}$, it can be seen that for any given $(k_1,\gamma)\in\Omega_{M,\beta,\epsilon_0}$, noting that $0<\gamma<1$, we have

$$\begin{aligned}&\frac{1}{\gamma}\bigg\{1+\frac{2(1+\beta)k_1d^*+1}{2}+\beta k_1 d^*\\&+\frac{(r(\mu,\beta\mu)(1+\beta))(1+2\beta)k_1^2\lambda_N d^*\sqrt{N}}{(\gamma-\rho(k_1,\beta k_1))}\bigg\}\\&\leq\frac{1}{\gamma}\bigg\{1+\frac{2(1+\beta)k_1d^*+1}{2\gamma}+\frac{\beta k_1 d^*}{\gamma^2}+(r(\mu,\beta\mu)(1+\beta))k_1^2\lambda_N d^*\\&\times\sqrt{N}\Big[\frac{1+\beta}{\gamma(\gamma-\rho(k_1,\beta k_1))}+\frac{\beta}{\gamma^2(\gamma-\rho(k_1,\beta k_1))}\Big]\bigg\}\\&\leq M+\frac{3}{2}\end{aligned}$$

which together with Lemma 4.1 gives

$$k_1\leq\frac{M(1-\epsilon_0)[(\beta-1)\lambda_2+o(1)]}{2(r_0+o(1))(1+\beta)(1+2\beta)\lambda_N d^*\sqrt{N})]}. \tag{A.18}$$

From the above, noting that $\lim_{k_1\to0}\rho(k_1,\beta k_1)=1$, we know that, there exists an integer $N_0(\beta, M)$, such that $\gamma=1-\epsilon_0(1-\rho(k_1,\beta k_1))\geq 1/(M+1/2)$, $N\geq N_0$, which together with $1-\epsilon_0(1-\rho(k_1,\beta k_1))\in(\rho(k_1,\beta k_1),1)$ and the definition of $\Omega_{M,\beta,\epsilon_0}$ gives $\Omega_{M,\beta}\supseteq\bigcup_{\epsilon_0\in(0,1)}\Omega_{M,\beta,\epsilon_0}$, $N\geq N_0$. Then it is easily verified that $\Omega_{M,\beta}\subseteq\bigcup_{\epsilon_0\in(0,1)}\Omega_{M,\beta,\epsilon_0}$. Hence, we have (A.17). Then by (A.17), (A.18) and Lemma 4.1 again, noting that $d^*\geq\lambda_N/2$, and $\sup_{\beta>1}(\beta-1)^2/(1+\beta)(1+2\beta)=1/2$, we have

$$\begin{aligned}&1-\inf_{\beta>1}\inf_{(k_1,\gamma)\in\Omega_{M,\beta}}\gamma\\&=1-\inf_{\beta>1}\inf_{\epsilon_0\in(0,1)}\inf_{k_1\in\Omega_{M,\beta,\epsilon_0}}(1-\epsilon_0(1-\rho(k_1,\beta k_1)))\\&=\sup_{\beta>1}\sup_{\epsilon_0\in(0,1)}\sup_{k_1\in\Omega_{M,\beta,\epsilon_0}}\epsilon_0(1-\rho(k_1,\beta k_1))\\&\leq\frac{M\lambda_2^2}{16r_0\lambda_N^2\sqrt{N}}+o\left(\frac{1}{\sqrt{N}}\right)\end{aligned}$$

which gives (40). □

Proof of Lemma 4.3: For any given integer $M\geq1$, constants $\beta_0>1$, $\epsilon_0\in(0,1)$, $\eta_0\in(0,1)$, $\delta_0\in(0,1)$ and $\tilde{\delta}_0\in(0,1)$, from Lemma 4.1, we know that there exists a constant $c_1(M,\epsilon_0,\eta_0,\delta_0,\tilde{\delta}_0)$, such that

$$\begin{cases}(1-\delta_0)M\leq M+(M+\frac{3}{2})[(1\\\quad-\epsilon_0(1-\rho(k_1,\beta_0k_1)))^3-1]\\1-\rho(k_1,\beta_0k_1))\leq\frac{1}{\epsilon_0}(1-\frac{2}{2M+1})\\\frac{(\beta_0-1)\lambda_2(1-\tilde{\delta}_0)k_1}{2}\leq1-\rho(k_1,\beta_0k_1)\\(1+\eta_0)r_0\geq r(k_1,\beta_0k_1)\\\forall\,k_1\in(0,c_1].\end{cases} \tag{A.19}$$

So there exists a positive integer $\bar{N}_0(\beta_0,\epsilon_0,M,\eta_0,\delta_0,\tilde{\delta}_0)$, such that

$$\begin{aligned}&\frac{M\lambda_2(\beta_0-1)}{(1+2\beta_0)(\beta_0-1)\lambda_2d^*+2(1+\eta_0)r_0(1+\beta_0)(1+2\beta_0)d^*\lambda_N\sqrt{N}}\\&\leq\min\{c_1,\frac{\beta_0-1}{\beta_0^2\lambda_N}\},\ \forall\,N\geq\bar{N}_0.\end{aligned} \tag{A.20}$$

Denote

$$k_1^*=\frac{(1-\epsilon_0)(1-\delta_0)(1-\tilde{\delta}_0)(\beta_0-1)M\lambda_2}{d^*\lambda_N[(1+2\beta_0)(\beta_0-1)+2(1+\eta_0)r_0(1+\beta_0)(1+2\beta_0)\sqrt{N}]}\quad N\geq\bar{N}_0. \tag{A.21}$$

From (A.20), we know that

$$k_1^*\in(0,\min\{c_1,\frac{\beta_0-1}{\beta_0^2\lambda_N}\}]. \tag{A.22}$$

Denote

$$\gamma^*=1-\epsilon_0(1-\rho(k_1^*,\beta_0k_1^*)). \tag{A.23}$$

From (A.19) and (A.22), we have

$$\gamma^*\geq\frac{1}{M+\frac{1}{2}} \tag{A.24}$$

and

$$\begin{aligned}&[(1+2\beta_0)(\beta_0-1)\lambda_2(1-\epsilon_0)(1-\tilde{\delta}_0)k_1^*d^*\\&\quad+22(1+\eta_0)r_0(1+\beta_0)(1+2\beta_0)k_1^*d^*\lambda_N\sqrt{N}]\\&\quad\times[(1-\epsilon_0)(\beta_0-1)\lambda_2(1-\tilde{\delta}_0)]^{-1}\\&\leq[k_1^*d^*\lambda_N[(1+2\beta_0)(\beta_0-1)(1-\epsilon_0)(1-\tilde{\delta}_0)\\&\quad+2(1+\beta_0)(1+2\beta_0)\sqrt{N}]]\\&\quad\times[(1-\epsilon_0)(\beta_0-1)\lambda_2(1-\tilde{\delta}_0)]^{-1}\\&\leq(1-\delta_0)M\leq M+(M+\frac{3}{2})[(\gamma^*)^3-1],\ N\geq\bar{N}_0\end{aligned}$$

which implies

$$\begin{aligned}&1+\frac{2(1+\beta_0)k_1^*d^*+1}{2}+\beta_0k_1^*d^*\\&+\frac{r_0(1+\eta_0)(1+\beta_0)(1+2\beta_0)(k_1^*)^2d^*\lambda_N\sqrt{N}}{(1-\epsilon_0)(1-\rho(k_1^*,\beta_0k_1^*))}\leq(M+\frac{3}{2})(\gamma^*)^3.\end{aligned}$$

From the above and $\gamma^* - \rho(k_1^*, \beta k_1^*) = (1-\epsilon_0)(1-\rho(k_1^*, \beta_0 k_1^*))$, we have

$$\begin{aligned}&\frac{1}{\gamma^*} + \frac{2(1+\beta_0)k_1^* d^* + 1}{2(\gamma^*)^2} + \frac{\beta_0 k_1^* d^*}{(\gamma^*)^3} \\ &\quad + r(k_1^*, \beta_0 k_1^*)(1+\beta_0)(k_1^*)^2 \lambda_N d^* \sqrt{N} \\ &\quad \times \left[\frac{1+\beta_0}{(\gamma^*)^2(\gamma^* - \rho(k_1^*, \beta_0 k_1^*))} + \frac{\beta_0}{(\gamma^*)^3(\gamma^* - \rho(k_1^*, \beta k_1^*))}\right] \\ &\quad \le M + \frac{3}{2}\end{aligned}$$

which together with (A.22), (A.23) and (A.24) leads to $(k_1^*, \gamma^*) \in \Omega_{M,\beta_0}, N \ge \bar{N}_0$. Then from (A.19), (A.21), (A.22), and (A.23), we have

$$\begin{aligned}&1 - \inf_{\beta>1} \inf_{(k_1,\gamma)\in\Omega_{M,\beta_0}} \gamma \\ &\quad \ge 1 - \gamma^* \ge \epsilon_0 \frac{(\beta_0 - 1)\lambda_2(1-\tilde{\delta}_0)k_1^*}{2} \\ &\quad = \frac{\epsilon_0(1-\epsilon_0)(1-\delta_0)(1-\tilde{\delta}_0)^2 M \lambda_2^2(\beta_0-1)^2}{2\lambda_N d^*[(1+2\beta_0)(\beta_0-1)+r_0(1+\eta_0)(1+\beta_0)(1+2\beta_0)\sqrt{N}]} \\ &\quad \ge \frac{\epsilon_0(1-\epsilon_0)(1-\delta_0)(1-\tilde{\delta}_0)^2 M \lambda_2^2(\beta_0-1)^2}{2\lambda_N^2[(1+2\beta_0)(\beta_0-1)+r_0(1+\eta_0)(1+\beta_0)(1+2\beta_0)\sqrt{N}]} \\ &N \ge \bar{N}_0\end{aligned}$$

which implies

$$\begin{aligned}&\liminf_{N\to\infty} \frac{1 - \inf_{\beta>1}\inf_{(k_1,\gamma)\in\Omega_{M,\beta}} \gamma}{1 - \exp\{-\frac{MQ_N^2}{16 r_0 \sqrt{N}}\}} \\ &\quad \ge \frac{8(1+\eta_0)(\beta_0-1)^2 \epsilon_0(1-\epsilon_0)(1-\delta_0)(1-\tilde{\delta}_0)^2}{(1+\beta_0)(1+2\beta_0)} \\ &\quad \forall\, \beta_0 > 1, \epsilon_0 \in (0,1), \eta_0 \in (0,1), \delta_0 \in (0,1), \tilde{\delta}_0 \in (0,1).\end{aligned}$$

Let $\epsilon_0 = 1/2$, $\beta_0 \to \infty$ and η_0, δ_0 and $\tilde{\delta}_0$ go to 0, then we have (41). □

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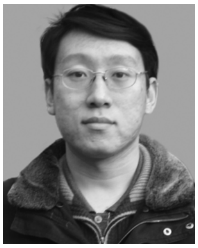

Tao Li (M'09) was born in Tianjin, China, in September, 1981. He received the B.S. degree in automation from Nankai University, Tianjin, in 2004, and the Ph.D. degree in systems theory from the Academy of Mathematics and Systems Science (AMSS), Chinese Academy of Sciences (CAS), Beijing, China, in 2009.

From 2008 to 2011, he held research positions in Nanyang Technological University. From September 2010 to January 2011, he was a Visiting Fellow of Australian National University. Since July 2009, he has been a faculty member of AMSS, CAS, where now he is an Assistant Professor. His current research interests include stochastic systems, networked control, multi-agent systems, and sensor networks.

Dr. Li was mentioned as one of the five finalists for the Young Author Prize of the 17th IFAC World Congress, 2008. He received the Special President Prize of Chinese Academy of Sciences in 2009, the Best Paper Award of the 7th Asian Control Conference with coauthors in 2009, the 2009 Singapore Millennium Foundation Research Fellowship, and the 2010 Endeavour Research Fellowship from Australian government. He is a member of the Youth Innovation Promotion Association, CAS. He is currently an editorial board member of Mathematical Problems in Engineering. Since 2010, he has been a program committee member of Chinese Control Conference.

Lihua Xie (S'91–M'92–SM'97–F'07) received the B.E. and M.E. degrees in electrical engineering from Nanjing University of Science and Technology in 1983 and 1986, respectively, and the Ph.D. degree in electrical engineering from the University of Newcastle, Callaghan, Australia, in 1992.

He held teaching appointments in the Department of Automatic Control, Nanjing University of Science and Technology, from 1986 to 1989. Since 1992, he has been with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore, where he is currently a Professor and Head of Division of Control and Instrumentation and Director, Centre for E-City. He was a Changjiang Visiting Professor with South China University of Technology from 2006 to 2010. His research interests include robust control and estimation, networked control systems, sensor networks, multi-agent systems, time delay systems and control of hard disk drive systems. He is an editor of IET Book Series on Control.

Dr. Xie has served as an Associate Editor of several journals including the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, *Automatica*, IEEE TRANSACTIONS ON CONTROL SYSTEMS TECHNOLOGY, IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS-II, and IET Proceedings on Control Theory and Applications. He is a Fellow of the IFAC.